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The Bell inequality: a measure of entanglement?

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Abstract. Entanglement is a critical resource used in many current quantum information schemes. As such entanglement has been extensively studied in two qubit systems and its entanglement nature has been exhibited by violations of the Bell inequality. Can the amount of violation of the Bell inequality be used to quantify the degree of entanglement? What do Bell inequalities indicate about the nature of entanglement?

Entanglement was recognized early as one of the key features of quantum mechanics [1, 2]. Entanglement can be described as the correlation between distinct subsystems which cannot be created by local actions on each subsystem separately. The advantage offered by quantum entanglement relies on the crucial premise that it cannot be reproduced by any classical theory [3–5]. Despite the fact that the possibility of quantum entanglement was acknowledged almost as soon as quantum theory was discovered, it is only in recent years that consideration has been given to finding methods to quantify it [6–17]. Historically the Bell inequalities were seen as a means of determining whether a two qubit system is entangled. It was known that the larger the violation of the Bell inequality the more the entanglement present in the system [18]. This led to the perception that to some degree the Bell inequalities were a measure of entanglement in such systems.

In 1994 it was discovered that not all entangled states violate a Bell inequality [19]. It was shown that the Werner state, a mixture of the maximally entangled state and the maximally mixed state can be entangled (inseparable) and yet not violate the conventional Bell inequality [20, 21]. It was found that multiple copies [22] of the Werner state could be distilled to a state that does violate the Bell inequality. Hence it is important to specify now that our interest lies in whether a single particular state violates such an inequality. It was shown by Gisin *et al*. [23] that there are states which do not violate this inequality, but can be distilled by local operations and classical communication to produce a state that does. Our interest is in whether the original state violates such inequalities. These observations are important experimentally because the Bell inequalities have been one of the few methods available to determine whether a two qubit state has quantum properties. There have been quite a number of experimental tests of the Bell

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inequality [24] using polarisation entangled photons from a spontaneous parametric down conversion source. Kwiat *et al.* [25] have shown a 242_{σ} violation of Bell's inequality. Here maximally entangled pure Bell states were produced, however, these sources are currently being used to synthesize two-qubit polarization quantum states, with a variable degree of entanglement and purity [26]. The question then becomes how do we characterize the entanglement in such systems. In the current work such states are being characterized by quantum state tomography which allows the reconstruction of the reduced density matrix for the polarization entangled photons [26, 27]. Hence one can determine all the physically relevant properties such as the degree of entanglement and purity. The process to reconstruct the density matrices requires many more measurements than those required to violate a Bell inequality.

This article is structured as follows. We begin by defining how to characterize a two qubit-state in terms of its degree of entanglement and degree of mixture. We then proceed to specify the particular Bell inequality considered here. At this point we comment again that we are interested in tests that can be performed on a single entangled pair of qubits (primarily because this is physically currently). There are a number of possible measurements that can be performed but here we restrict our attention to POVMs (positive operator valve measure). We will not consider generalized measurements. Given these tools we now examine the degree of violation versus the degree of entanglement for two classes of states; the Werner state [20] and the maximally entangled mixed state [28]. We will attempt to answer the question as to 'what the Bell inequality indicates about the nature of entanglement?'. Is it only weakly entangled states that do not violate such inequalities?

In examining the degree of entanglement there are currently a number of measures available. These include the entanglement of distillation [6], the relative entropy of entanglement [29] but the canonical measure of entanglement is called the entanglement of formation (EOF) [6] and for a pure state is simply given by the von Neumann entropy [30] of that reduced density matrix. For a mixed state $\hat{\rho}$ the entanglement of formation is defined as,

$$E_F(\hat{\rho}) = \min \sum_i p_i E_F(\psi), \tag{1}$$

where this minimum is taken over all the possible decompositions of $\hat{\rho}$ into the pure states $\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. The entanglement of formation for an arbitrary two-qubit system has been found by Wootters [13] to be simply given by

$$E_F(\hat{\rho}) = h\left(\frac{1+\sqrt{1-\tau}}{2}\right), \tag{2}$$

where $h(x)$ is Shannon's entropy function,

$$h(x) = -x \log(x) - (1-x)\log(1-x), \tag{3}$$

and τ is the tangle [31] (concurrence [15] squared). The tangle τ is given by,

$$\tau = C^2 = [\max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}]^2. \tag{4}$$

where the λs are the square root of the eigenvalues in decreasing order of,

$$\hat{\rho}\tilde{\rho} = \hat{\rho}\, \sigma_y^A \otimes \sigma_y^B \hat{\rho}^* \sigma_y^A \otimes \sigma_y^B. \tag{5}$$

Here $\hat{\rho}^*$ denotes the complex conjugation of $\hat{\rho}$ in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. With the entanglement of formation E_F being a strictly monotonic function of τ, the maximum of τ corresponds to the maximum of E_F. Hence the tangle may also be considered a direct measure of the degree of entanglement and this is what we will consider in this article. In general the measure of entanglement to be used depends heavily on the desired use of that information. The entanglement of distillation may be a much more useful practical measure but it is different to calculate in practice. In general the entanglement of distillation is smaller than entanglement of formation.

For a general two-qubit density matrix the purity of the state provides complementary information about the state. The purity measure described here is the linearized entropy [32] of $\hat{\rho}$ given by

$$S_L = \frac{4}{3}\{1 - \mathrm{Tr}[\rho^2]\}. \tag{6}$$

The 4/3 normalization [26] for S_L ensures that for a general two-qubit density matrix S_L ranges between 0 and 1. The von Neumann entropy [30] of the state could be used but S_L is easier to calculate and provides the same degree of characterization. With an explicit expression for the degree of entanglement and the degree of mixture let us now turn our attention to the Bell inequality and what a violation of it potentially indicates about the nature of entanglement for the two qubit system. There are a large number of Bell inequalities that could be investigated in this article but we will focus our attention on the *original* two-qubit Bell inequality [3-5],

$$\mathbf{B}_\mathrm{S} = \left|E(\phi_1, \phi_2) + E(\phi_1 \phi_2') + E(\phi_1', \phi_2) + E(\phi_1', \phi_2')\right| \leq 2, \tag{7}$$

where the correlation function $E(\phi_1, \phi_2)$ is given by,

$$E(\phi_1, \phi_2) = \mathrm{Tr}\{\hat{S}_1(\phi_1)\hat{S}_2(\phi_2)\hat{\rho}\}, \tag{8}$$

with,

$$\hat{S}_i(\phi_i) = \cos\phi_i[|0\rangle\langle 0| - |1\rangle\langle 1|] + \sin\phi_i[|0\rangle\langle 1| + |1\rangle\langle 0|]. \tag{9}$$

The inequality (7) is violated if $\mathbf{B}_\mathrm{S} > 2$. In the above expression the ϕ_is are the analyser settings for the i^{th} particle ($i = 1, 2$). In calculating whether the Bell inequality is violated, the choice of analyser settings is absolutely critical. In this article we will choose them to maximize the violation for the actual state under consideration.

It is now time to turn attention to the class of states considered in this article. The Hilbert space in which two qubit reside is large and hence in this article we will generally restrict our attention to two particular types of states. The first state is of the form,

$$\hat{\rho}(\gamma, \xi) = \frac{1-\gamma}{4} I_2 \otimes I_2 + \gamma|\Psi_\mathrm{non}\rangle\langle\Psi_\mathrm{non}|, \tag{10}$$

where,

$$|\Psi_\mathrm{non}\rangle = \cos\xi|0\rangle|0\rangle + e^{i\phi}\sin\xi|1\rangle|1\rangle. \tag{11}$$

For $\xi = \pi/4$ eqn (10) is of the form normally attributed to the usual Werner state [20] which was the first state found to be entangled for certain γ and yet not violate a Bell inequality for single states.

We will refer to the state (10) as a non-maximal Werner state as it is a mixture of a non-maximally entangled state and the fully mixed state. In the limit of $\gamma = 1$ equation (10) represents a non-maximally entangled pure state. It is straight forward to show that the mixture of the non-maximally entangled state and the fully mixed state given by (10) is entangled only when,

$$\gamma > \frac{1}{1 + 2|\sin(2\xi)|} \tag{12}$$

(for the Werner state it is entangled for $\gamma > \frac{1}{3}$ [8, 33]). It is also possible to derive an explicit expression for the degree of entanglement for such states using the tangle measure. While it is quite complicated one can show that the tangle for the state (10) is given by,

$$\tau = \left[\max\{\sqrt{\Lambda_1} - \sqrt{\Lambda_2} - \frac{\gamma - 1}{2}, 0\}\right]^2, \tag{13}$$

where,

$$\Lambda_{\frac{1}{2}} = \pm 4\gamma|\sin(2\xi)|\sqrt{(1+\gamma)^2 - 4\gamma^2\cos^2(2\xi)} + (1+\gamma)^2 - 4\gamma^2\cos(4\xi). \tag{14}$$

The second state considered is the maximally entangled mixed state recently predicted by White *et al.* [26]. This states has the explicit form,

$$\hat{\rho}_{mems} = \begin{pmatrix} g(\gamma) & 0 & 0 & \frac{\gamma}{2} \\ 0 & 1 - 2g(\gamma) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\gamma}{2} & 0 & 0 & g(\gamma) \end{pmatrix}, \tag{15}$$

where,

$$g(\gamma) = \begin{cases} \gamma/2 & \gamma \geq 2/3 \\ 1/3 & \gamma < 2/3 \end{cases}, \tag{16}$$

and has been shown to have the maximal amount of entanglement for a certain degree of mixture (as measured by linear entropy) or vice versa. This state is entangled for all nonzero γ and in fact it has been shown that the tangle simply given by

$$\tau = \gamma^2. \tag{17}$$

For a given degree of mixture, the maximally entangled mixed state is generally significantly more entangled that the Werner state at the same degree of mixture.

Let us now examine how well these two state violate a Bell inequality. The state (10) violates the Bell inequality (7) for quite a wide range of γ, ξ values. In figure (1) the maximum value of $\mathbf{B}_S$ (optimizing the analyser settings to maximize the violation) is presented against the degree of entanglement (as measured by the tangle). Two specific parameter sets are plotted

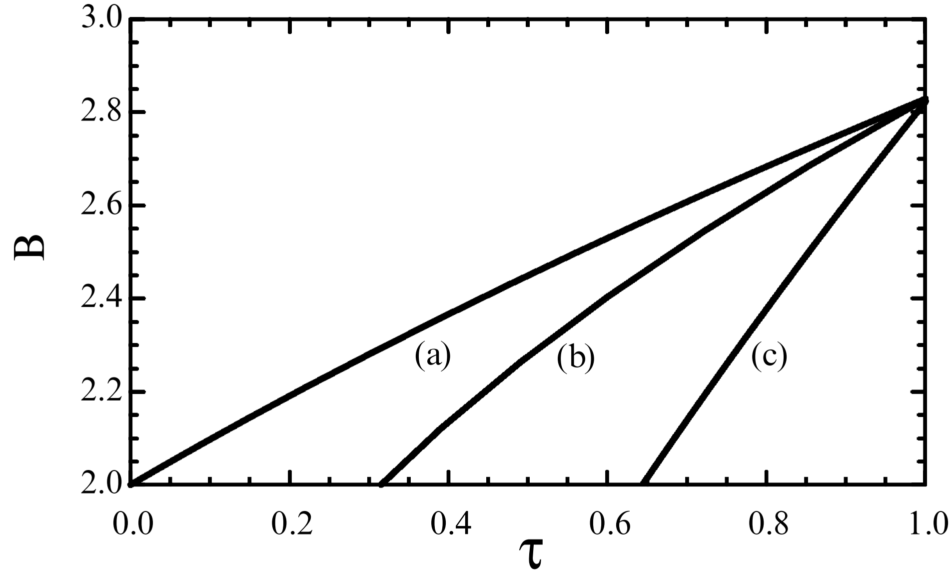

Figure 1. Plot of the maximum violation of the spin Bell inequality versus the degree of entanglement (tangle τ) for the non-maximally entangled pure state represented by the density operator $\hat{\rho}(1,\xi)$ (curve a) and the Werner state represented by the density operator $\hat{\rho}(\gamma,\pi/4)$ (curve b). The analyser settings have been chosen to maximize the violation for the particular γ,ξ values. A violation of the spin Bell inequality is achieved when $|B_S|>2$. Also shown in the figure are the results for the maximally entangled mixed state (curve c).

- the non-maximally entangled pure state $\hat{\rho}(1,\xi)$;
- and the Werner state given $\hat{\rho}(\gamma,\pi/4)$.

This result shows very clearly that the Werner state (given by the density matrix $\hat{\rho}(\gamma,\pi/4)$) and the non-maximally entangled pure state (given by the density matrix $\hat{\rho}(1,\xi)$) violate the Bell inequality by different amounts for the same degree of entanglement. In fact for these two different classes of entangled states, there is a clear region where one of the states (the non-maximally entangled pure state) violates the Bell inequality but the Werner state does not [19]. This was a surprising result when it was first found by Popescu [19]. It showed that not all entangled states violate a Bell inequality. All pure two-qubit entangled states do violate a Bell inequality [34] and in fact the degree of violation is equal to $2\sqrt{1+\tau^2}$ [35]. However as a state becomes mixed it is harder to violate the Bell inequality. The Werner state does not violate our Bell inequality if its tangle is less than $\tau \leq 1/3$ ($EOF = 0.44229$). This is quite a small degree of entanglement and has led to the perception that it is only certain weakly entangled mixed states that do not violate the two-qubit Bell inequality. To investigate this point further consider the maximally entangled mixed state described in (15). This state has it degree of entanglement maximized for a given purity and vice versa.

In figure (1), curve c plots the degree of violation of the Bell inequality versus the tangle for this maximally entangled mixed state. It is observed in this figure that significantly more entanglement is required to violate the Bell inequality to the same degree for the maximally entangled mixed state than for the Werner state. In fact our Bell inequality for the maximally entangled mixed state is only violated if $\tau > 0.64$ ($EOF = 0.721928$). This is a significant degree of entanglement given that a Bell state has $\tau = 1.0$ ($EOF = 1.0$) and a separable state has $\tau = 0.0$ ($EOF = 0.0$). The maximally entangled mixed state considered here has a maximal degree of entanglement for a given linear entropy (the choice of degree of mixture in this case). There are other choices for the degree of mixture, not based on purity, and these may have a tangle value $\tau > 0.64$ while still not violating Bell inequality. This is left for further investigation.

The result above also tentatively indicates that the more mixture contained in a state, the higher the degree of entanglement required for it to violate the two-qubit Bell inequality. These results indicate that if a state has a certain degree of entanglement (this may be large), it is not possible to infer whether that state will violate the Bell inequality or by how much. This is, we believe, the first instance where it has been explicitly demonstrated via quantifiable measures that the size of the violation of the Bell inequality for an unknown two-qubit state is not absolutely related to the degree of entanglement in that state.

Let us now investigate in some detail the effect of mixture of our entanglement and the Bell inequality question. Again we will examine two specific states, the first being the non-maximally entangled Werner state. We choose this state as it has the property that with the two parameters γ, ξ we can change the state from a non-maximally entangled pure state to the Werner state. It is known that the non-maximally entangled pure state $\gamma = 1$ violates the Bell inequality as soon as the state contains entanglement ($\xi \neq 0$)[34]. However, the Werner state (with $\xi = \pi/4$ violates the Bell inequality only when $\tau > \frac{1}{3}$. We will investigate what occurs between these two extremes. The second state examined is a modification of the maximally entangled mixed state,

$$\hat{\rho}_m(\gamma, \xi) = (1 - \gamma)|0\rangle|1\rangle\langle 0|\langle 1| + \gamma|\Psi_{\mathrm{non}}\rangle\langle\Psi_{\mathrm{non}}|, \tag{18}$$

and is a mixture of the non-maximally entangled pure state and the diagonal density matrix element $|0\rangle|1\rangle\langle 0|\langle 1|$. As for the first state mentioned the two parameters in this state also make it possible to change its behaviour for a non-maximally entangled pure state to the maximally entangled mixed state. Choosing the parameters γ and ξ such that both states (10) and (18) are a non-maximally entangled pure states that just satisfy the Bell inequality (that is $\mathbf{B}_S = 2$) we vary the parameters γ, ξ such that the degree of mixture is increased in the system while maintaining $\mathbf{B}_S = 2$. For these new γ and ξ values the degree of entanglement and mixture is then determined ensuring that $\mathbf{B}_S = 2$.

In figure (2) on the tangle-linear entropy plane, the boundary curve is plotted where $\mathbf{B}_S = 2$ for both states. The tangle axis (y-axis) represents the degree of entanglement while the x-axis displays the degree of mixture. Figure (2) confirms for these states the idea that as the state becomes more mixed, more entanglement is required to violate the Bell inequality. If we again examine the state (10) then points for this state that fall below the curve (a) in figure (2) are entangled (if $\tau > 0$) but do not violate the inequality considered.

To summarize, in this article we have investigated the extent to which the Bell inequality may be considered a measure of entanglement. Results indicate that the more mixed a system is made the more entanglement is generally required to violate the original Bell inequality to the same degree. We have specifically

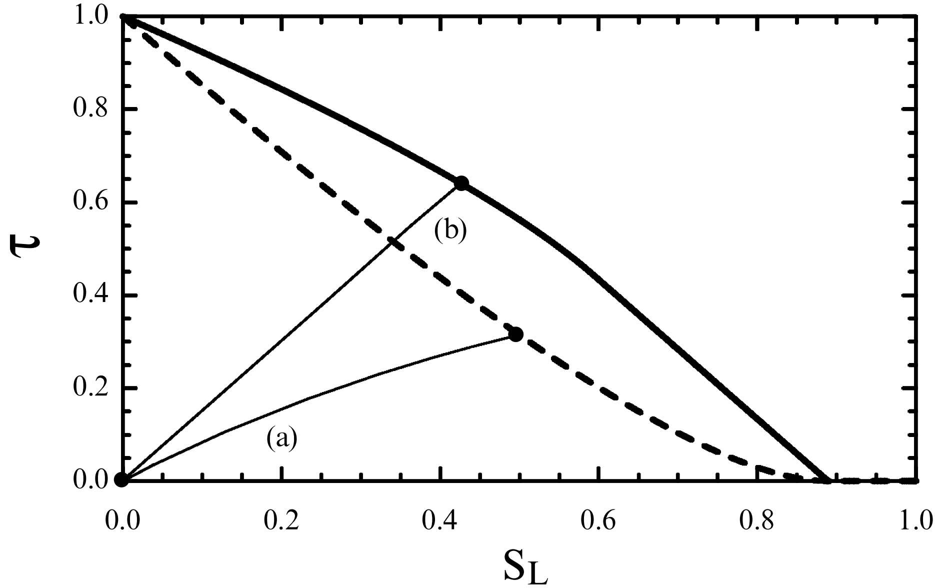

Figure 2. Plot of the degree of entanglement versus linear entropy for the states (10) and (15). The Werner state is displayed as a dotted line while the MEMS state (15) is displayed as the solid dark curve. The tangle τ of the Werner and MEMS state increases as the system becomes less mixed. The non-maximally entangled pure state may be represented by a line along the y-axis at a linear entropy of $S_L = 0$. The non-maximally entangled pure state satisfies the Bell inequality ($\mathbf{B}_S = 2$) at $\tau = 0$. Curve (a) traces out the curve for the state (10) where γ and ξ are chosen such that $\mathbf{B}_S = 2$. Curve (b) traces out the curve for the state (18) where γ and ξ are chosen such that $\mathbf{B}_S = 2$. In all situations here the analysers setting were chosen to maximize the potential violation.

illustrated an example where a state (the maximally entangled mixed state) has a tangle of $\tau = 0.64$ which represents a significant degree of entanglement (an $EOF = 0.72$) yet does not violate the Bell inequality considered here. This dispels the impression that it is only weakly entangled states that do not violate the Bell inequality. For a specific class of state, for instance the Werner state, it is clear that as the degree of entanglement increases, so does $\mathbf{B}_{\max}$ and hence the potential violation. However, without full knowledge of the state being analysed and given that the two-qubit state has a certain degree of entanglement, it is impossible generally to determine the extent to which the Bell inequality is violated (or for a small degree of entanglement if it is violated). In terms of finding other more generalized Bell inequalities that are violated, the maximally entangled mixed state is a good test candidate. To finish, however, the knowledge that the Bell inequality is violated is strong evidence for the presence of entanglement in the two-qubit system. The Bell inequality can always be seen as a *marker* for entanglement.

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